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NATURAL FREQUENCY OF SKEW PLATES USING FIRST-ORDER SHEAR DEFORMATION THEORY

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ABSTRACT

This paper presents the free vibration analysis of skew plates based on the first-order shear deformation theory (FSDT). The development of finite element plates based on first-order shear deformation plate theory has been carried out and provides good results in plate element analysis. In this study, we investigate plate analysis in the case of free vibration to obtain natural frequency using one of the plate elements developed based on FSDT, numerical analysis was performed on skew plates case with varying skew angles and length to thickness ratios, the result will be used to see the convergence behavior and performance of plate element by comparing with the reference solution in the literature.

Keywords: *Free vibration; Natural frequency; First-order shear deformation theory; Finite element.*

1. PRELIMINARY

The numerical method is one of the tools used to analyze plate elements, the finite element method is part of the numerical method that is often involved in analyzing plate elements. In the finite element plate analysis, the main thing to be achieved is how to generate plate elements that have high accuracy and convergence rate, fast computation time, and can be applied in various conditions. Shear locking is a challenge in developing plate elements that have good performance, this phenomenon occurs when length to thickness ratios (L/h) increase. Shear locking arises because the plate cannot qualify as a Kirchoff plate when the length to thickness ratios is increased, under these conditions the shear effect is not reduced when the plate becomes thin, the dependence of the plate element on its thickness will cause shear locking which results in inaccurate plate analysis results.

Many researchers propose approaches to analyze plate elements, one of the concepts is the first-order shear deformation plate theory (FSDT), MITC3 (3-node triangular mixed interpolation of tensorial components) which was developed by Lee & Bathe **Error! Reference source not found.**]

is a plate element which was developed based on FSDT. MITC3 plate element was developed through the ''mixed interpolation of tensorial components'' (MITC) concept by Dvorkin and Bathe **Error! Reference source not found.-Error! Reference source not found.**], plate elements based on the MITC concept have successfully overcome shear locking in the case of square elements. The use of triangular elements attracts many researchers because triangular elements are the most efficient elements in discretizing elements, especially complex elements which are the weakness of square elements, one of the plate elements that uses a triangular element is MITC3.

Many studies on plate elements using MITC3 elements, especially in static cases, and research on improving the performance of MITC3 elements have also been conducted **Error! Reference source not found.-Error! Reference source not found.**]. The research plate element using finite element in the case of free vibration have been carried out by several researchers and get good results **Error! Reference source not found.Error! Reference source not found.Error! Reference source not found.**]. In this paper we conducted free vibration analysis in the case of the skew plate, varying length to thickness ratios and skew angle, to see the effect of thickness and skew angles on the results of plate analysis, the vibration analysis is investigated using MITC3 elements to get the natural frequency through eigenvalue problem. The numerical analysis results will be compared with the free vibration analysis result using other elements or exact solutions, from the comparison it can be seen the level of convergence and performance of the MITC3 elements.

2. THEORETICAL BASIS

Strain-Displacement Equation

According to Reissner-Mindlin plate first-order shear deformation plate theory **Error! Reference source not found.**], the displacement field can be declared as:

$$\begin{aligned} u(x,y,z) &= u(x,y) + z\beta_x(x,y) \\ v(x,y,z) &= v(x,y) + z\beta_y(x,y) \\ w(x,y,z) &= w(x,y) \end{aligned} \qquad (1)$$

where *u*, *v*, *w* are displacements of the mid-plane of the plate, β_x and β_y represent the rotations of the transverse normal about the x and y axes, respectively.

The linear strains are given by:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y} + \beta_{y,x} \end{Bmatrix} = z\{\chi\} \qquad (2)$$

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \beta_x + w_{,x} \\ \beta_y + w_{,y} \end{Bmatrix} = \{\gamma\} \qquad (3)$$

The notation of $\beta_{x,x}$ and $\beta_{y,y}$ state the first derivatives concerning *x* and *y* respectively of β_x. $w_{,x}$ and $w_{,y}$ are the first derivatives concerning *x* and *y* respectively of vertical displacement *w*.

Constitutive Equations

According to hook law's the stress in the plane can be declared as:

$$\{\sigma\} = [E]\{\varepsilon\} \qquad (4)$$

The shear stress is as follows:

$$\{\tau\} = [G]\{\gamma\} \qquad (5)$$

The constitutive equations can be simplified as:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \dfrac{(1-v)}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} \quad ; \quad \begin{Bmatrix}\tau_x\\ \tau_y\end{Bmatrix} = \frac{E}{2(1+v)}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} \qquad (6)$$

Where the matrix of material is:

$$[\mathrm{E}] = \frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \dfrac{(1-v)}{2}\end{bmatrix} \quad ; \quad [\mathrm{G}] = \frac{k\ E}{2(1+v)}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad (7)$$

The notation of E declares the modulus of elasticity, v is Poisson's ratio, and k= 5/6 is the shear correction factor.

3. RESULT AND DISCUSSION

The Formulation of MITC3 Plate Element

MITC3 is the element proposed by Lee and Bathe **Error! Reference source not found.**], this element is based on the concept "mixed interpolation of tensorial components" (MITC) by Dvorkin and Bathe **Error! Reference source not found.-Error! Reference source not found.**] which uses the tying points to get shear strain matrix.

MITC3 element has 3 nodes with 3 degrees of freedom for each node, namely: w_i (translation in the z-direction) β_{xi} (rotation in the z-x plane), and β_{yi} (rotation in the z-y plane).

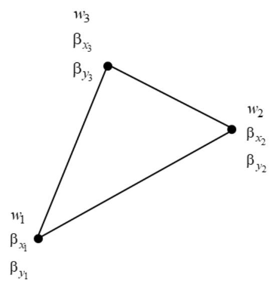

Figure 1. MITC3 elements with 3 nodes and 3 dof per node.

Independent rotation field β_x and β_y and displacement w are declared as:

$$w = \sum_{i=1}^{3} = N_i w_i \qquad \beta_x = \sum_{i=1}^{3} = N_i \beta_{x_i} \qquad \beta_y = \sum_{i=1}^{3} = N_i \beta_{y_i} \qquad (8)$$

Where N_i is the linear shape function at node-i.

$$N_1 = 1-\xi-\eta \quad ; \quad N_2 = \xi \quad ; \quad N_3 = \eta \qquad (9)$$

The Bending Strain for MITC3

The relationship between nodal variables and curvature $\{\chi\}$ is:

$$\{\chi\} = [B_b]\{u_n\}$$

$$[B_b] = \begin{bmatrix} 0 & N_{i,x} & 0 \\ \dots 0 & 0 & N_{i,y} \dots \\ 0 & N_{i,y} & N_{i,x} \end{bmatrix}_{i=1,2,3} \quad (10)$$

The Shear Strain for MITC3

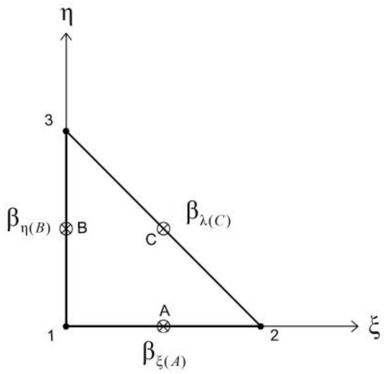

Figure 2. Tying point

The tying point is chosen in the mid-points of sides 1-2, 1-3, and 2-3. Distribusi β_ξ is assumed constant ξ and β_η is assumed constant along η.

$$\beta_\xi = a_1 + a_2\eta \quad ; \quad \beta_\eta = b_1 + b_2\xi \quad ; \quad \beta_\lambda = \frac{1}{\sqrt{2}}(\beta_\xi - \beta_\eta) \quad (11)$$

The value of β_ξ and β_η, at the tying points, are the average of the values for each corner of the side, then:

$$\begin{aligned} \beta_{\xi(A)} &= \frac{1}{2}\left(\beta_{\xi_1} + \beta_{\xi_2}\right) \;;\; \beta_{\eta(B)} = \frac{1}{2}\left(\beta_{\eta_1} + \beta_{\eta_3}\right) \\ \beta_{\xi(C)} &= \frac{1}{2}\left(\beta_{\xi_2} + \beta_{\xi_3}\right) \;;\; \beta_{\eta(B)} = \frac{1}{2}\left(\beta_{\eta 3} + \beta_{\eta_3}\right) \end{aligned} \quad (12)$$

Hence, the shear strain matrix is as below;

$$\begin{aligned} [B_s] &= [j]\left[B_{s_\xi}\right] = \left[\left[B_{s_1}\right] \; \left[B_{s_2}\right] \; \left[B_{s_3}\right]\right] \\ \left[B_{s_1}\right] &= \frac{1}{4A}\begin{bmatrix} -y_{13} & -y_{21} \\ x_{13} & x_{21} \end{bmatrix}\begin{bmatrix} -2 & (x_{21} + x_{32}\eta) & (y_{21} + y_{32}\eta) \\ -2 & -(x_{13} + x_{32}\xi) & -(y_{13} + y_{32}\xi) \end{bmatrix} \\ \left[B_{s_2}\right] &= \frac{1}{4A}\begin{bmatrix} -y_{13} & -y_{21} \\ x_{13} & x_{21} \end{bmatrix}\begin{bmatrix} 2 & (x_{21} + x_{32}\eta) & (y_{21} + y_{32}\eta) \\ 0 & -x_{13}\xi & -y_{13}\xi \end{bmatrix} \\ \left[B_{s_3}\right] &= \frac{1}{4A}\begin{bmatrix} -y_{13} & -y_{21} \\ x_{13} & x_{21} \end{bmatrix}\begin{bmatrix} 0 & x_{21}\eta & y_{21}\eta \\ 2 & -(x_{13} + x_{21}\xi) & (y_{13} + y_{21}\xi) \end{bmatrix} \end{aligned} \quad (13)$$

The Stiffness Matrix of MITC3

The total energy due to bending and shear can be stated as:

$$\Pi_{\text{int}} = \Pi_{\text{int}}^b + \Pi_{\text{int}}^s \quad (14)$$

Where Π_{int}, Π_{int}^b, and Π_{int}^s are internal, bending, and shear energy respectively.

The total stiffness due to bending, and shear can be expressed as:

$$[k] = [k_b] + [k_s] \quad (15)$$

Where

$$[k_b] = \int_A [B_b]^T [H_b][B_b]dA \quad ; \quad [k_s] = \int_A [B_s]^T [H_s][B_s]dA \quad (16)$$

Free Vibration Analysis

In buckling analysis, the thing to be achieved is to get a natural frequency value ω due to a given mass and get a mode shape. To get these two variables, it can be solved through the eigenproblem **Error! Reference source not found.**]:

$$\left([k]-\omega_n^{2}[m]\right)\{d\}=\{0\} \quad (17)$$

Where

$[k]$ is structural stiffness matrix, ω_n is natural frequency, $[m]$ is a mass matrix, and $\{d\}$ is mode shape.

Mass Matrix

The external energy equation due to free vibrations on the plate is:

$$\Pi_{ext}=\langle u_n\rangle[m]\{u_n\} \quad (18)$$

Where is the nodal displacement vector $\langle u_n\rangle$ as follows:

$$\langle u_n\rangle=\langle w_1 \quad \beta_{x_1} \quad \beta_{y_1} \quad w_2 \quad \beta_{x_1} \quad \beta_{y_2} \quad w_3 \quad \beta_{x_3} \quad \beta_{y_3}\rangle \quad (19)$$

The mass matrix in the free vibration analysis of the plate can be expressed as:

$$[m]=[m_w]+\left[m_{\beta_x}\right]+\left[m_{\beta_y}\right] \quad (20)$$

The $[m_w]$ matrix is a mass matrix of displacement in the z-direction.

$$[M_w]=\rho_m\int_A\{N_w\}\langle N_w\rangle dA \quad (21)$$

The $\left[m_{\beta_x}\right]$ and $\left[m_{\beta_y}\right]$ matrices are the mass matrix of rotation in the x and y directions.

$$\left[M_{\beta_x}\right]=\rho_b\int_A\left\{N_{\beta_x}\right\}\left\langle N_{\beta_x}\right\rangle dA \quad ; \quad \left[M_{\beta_y}\right]=\rho_b\int_A\left\{N_{\beta_y}\right\}\left\langle N_{\beta_y}\right\rangle dA \quad (22)$$

The mass matrix contains density which can be calculated using the following equation

$$\rho_m=\rho(z)dz \quad ; \quad \rho_b=\rho(z)z^2dz \quad (23)$$

The notation of ρ_m, and ρ_b declare the mass density related to the membrane and bending, N_w, N_{β_x}, and N_{β_y}, are shape function related to deflection and rotation respectively.

Numerical Analysis

Numerical analysis was performed on the fully modeled skew plate. Plate elements with boundary conditions hard simply supported (w=0, and β_x=0) on AB and CD sides (Figure 3), and clamped (w=0, β_x=0, and β_y=0) on AD and BC sides. The plate has a mesh size $N \times N \times 2$ with N= 4, 8, 16, 32, 64, and 128, these elements were analyzed at several skew angles of 30°, 45°, 60°, and 75°, and have two lengths to thickness ratios of (L/h=5), and (L/h=1000). Validating the convergence behavior plate, the reference of natural frequency is $\underline{\omega}=\left(\omega a^2/\pi^2\right)\sqrt{\rho h/D}$ is given by Woo et al. **Error! Reference source not found.**].

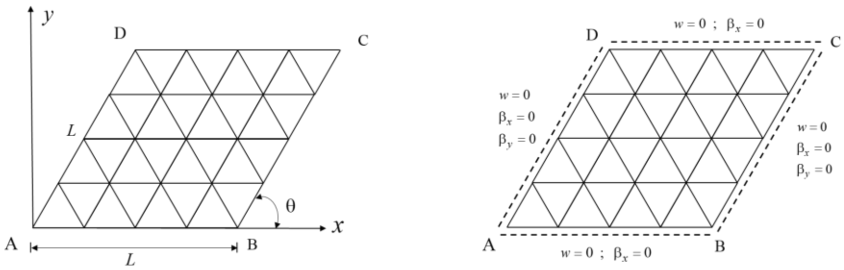

Figure 3. The modeling of the Rectangular plate with SCSC boundary conditions

Table 1 The first five non-dimensional natural frequency $\underline{\omega}$ of SCSC skew plate with L/h=5.

$N \times N \times 2$	Mode number	Skew Angle θ			
		$\theta = 30^\circ$	$\theta = 45^\circ$	$\theta = 60^\circ$	$\theta = 75^\circ$
128 × 128 × 2	1	5.4610	3.5596	2.7372	2.3681
	2	6.4304	4.6202	3.7811	3.3818
	3	7.4230	5.3542	4.4683	4.1095
	4	9.0747	7.0675	5.6500	4.7866
	5	9.1981	7.2253	6.2014	5.8832
Woo et al. **Error! Reference source not found.**]	1	5.5062	3.5683	2.7336	2.3605
	2	7.4286	5.3506	4.4573	4.0975
	3	9.1025	7.0200	5.6505	4.7792
	4	10.875	7.4414	6.1650	5.8785
	5	11.344	8.8146	7.9596	6.9815

Table 2 The non-dimensional natural frequency $\underline{\omega}$ of SCSC skew plate with L/h=5.

$N \times N \times 2$	Mode number	Skew Angle θ			
		$\theta = 30^\circ$	$\theta = 45^\circ$	$\theta = 60^\circ$	$\theta = 75^\circ$
4 × 4 × 2	1	5.0891	3.8324	3.0922	2.7707
8 × 8 × 2	1	5.4217	3.6004	2.8130	2.4592
16 × 16 × 2	1	5.4167	3.5580	2.7525	2.3898
32 × 32 × 2	1	5.4366	3.5549	2.7398	2.3731
64 × 64 × 2	1	5.4522	3.5575	2.7375	2.3691
128 × 128 × 2	1	5.4610	3.5596	2.7372	2.3681
Woo et al. **Error! Reference**	1	5.5062	3.5683	2.7336	2.3605

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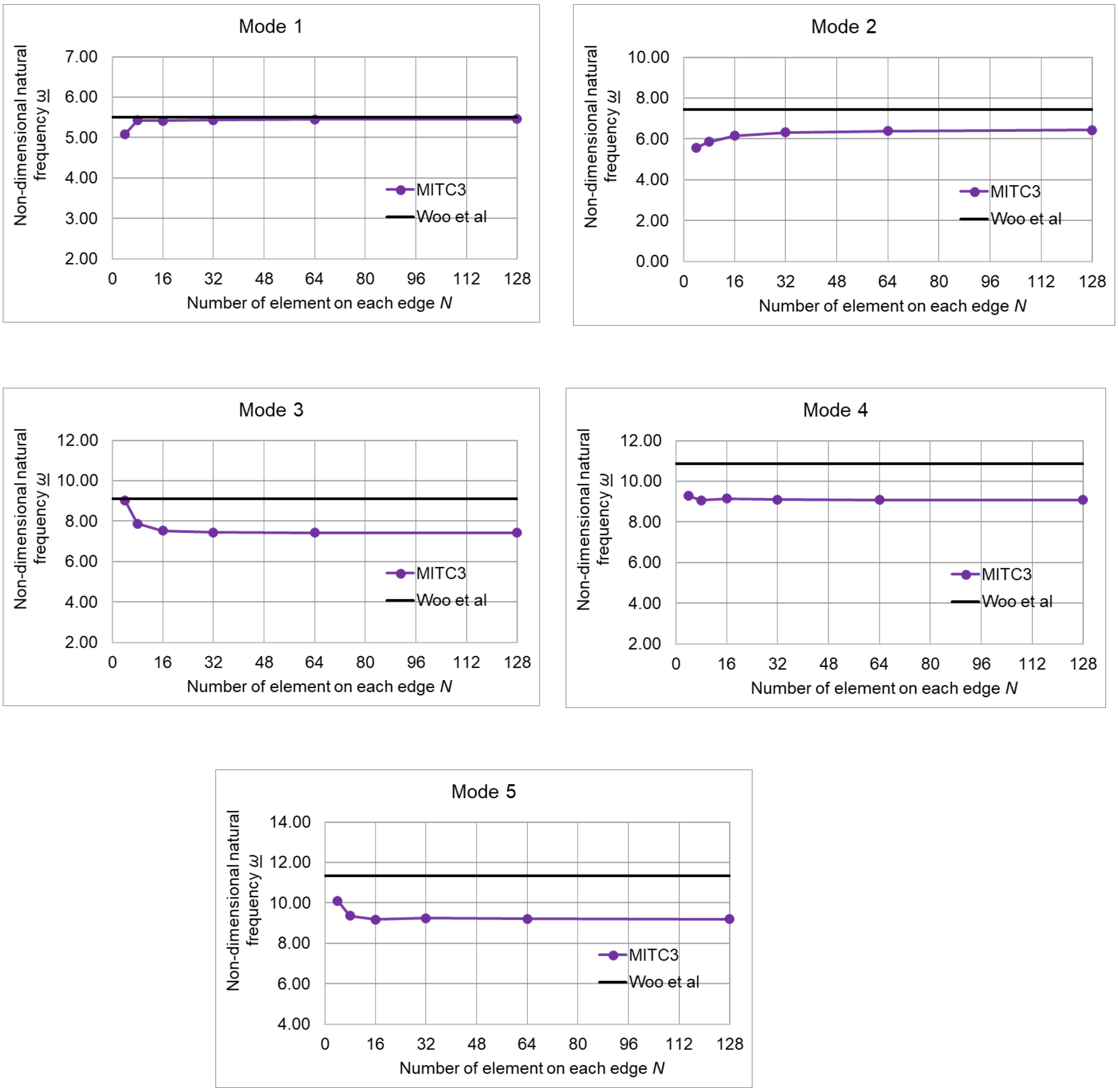


Figure 4. Nn-dimensional natural frequency $\underline{\omega}$ of SCSC skew plate (L/h = 5) with skew angle θ = 30.

The analysis in Table 1-2, noted that the natural frequency of plate with ratio L/h = 5 is convergence to the reference solution. Plate with mesh 4 × 4 × 2 until mesh 128 × 128 × 2, the difference in mode 1 skew angle θ = 30° to reference solutions is 0.821% - 7.576%, skew angle θ = 45° is 0.243% - 7.410%, skew angle θ = 60° is 0.133% - 13.118%, and skew angle θ = 75° is 0.324% - 17.377%. Figure 4 shows that at θ = 30 when the mesh is coarse, the largest difference with the reference solution occurs in mode 5, while the largest difference in the fine mesh occurs in mode 4.

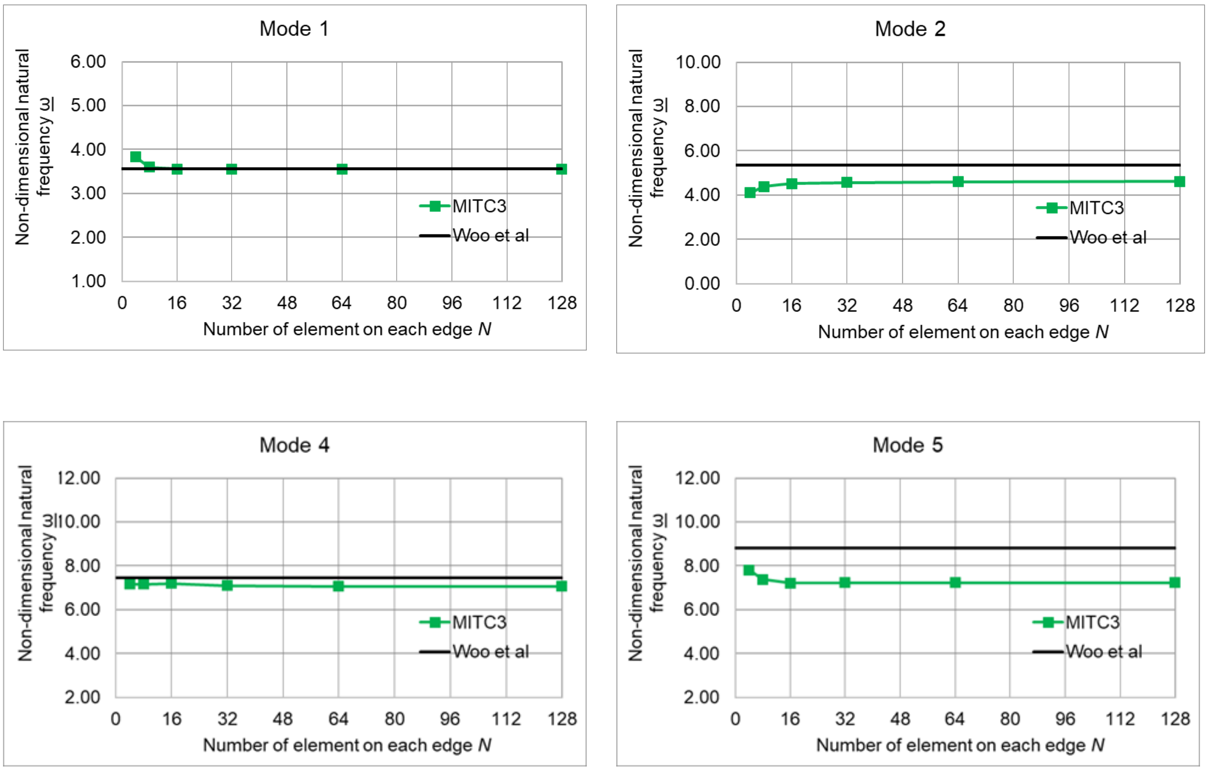
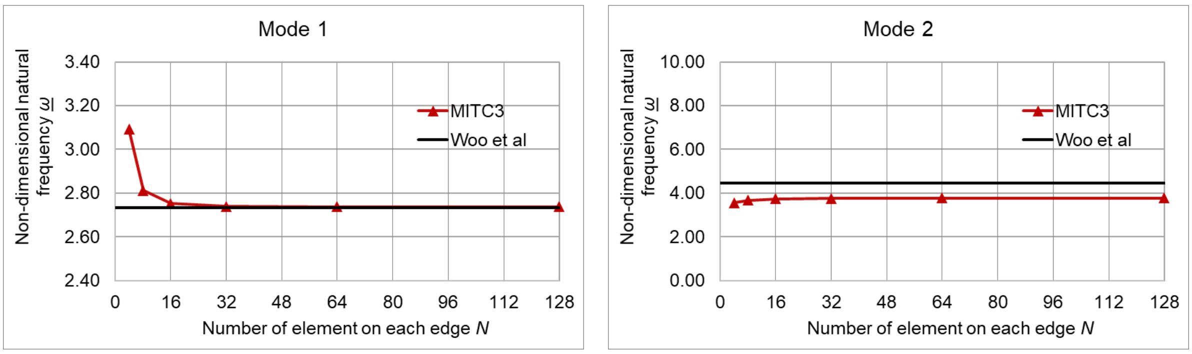


Figure 5. Non-dimensional natural frequency $\underline{\omega}$ of SCSC skew plate (L/h = 5) with skew angle θ = 45.

The natural frequency of skew plate with skew angle θ = 45˚ is smaller than a plate with skew angle θ = 30 of Figure 5. in fine mesh the smallest difference to the reference solution occurs in mode 1 which is 0,243% and the biggest difference is 23.370% in mode 3, the natural frequency difference between θ = 45˚ and θ = 30˚ is 1.901.



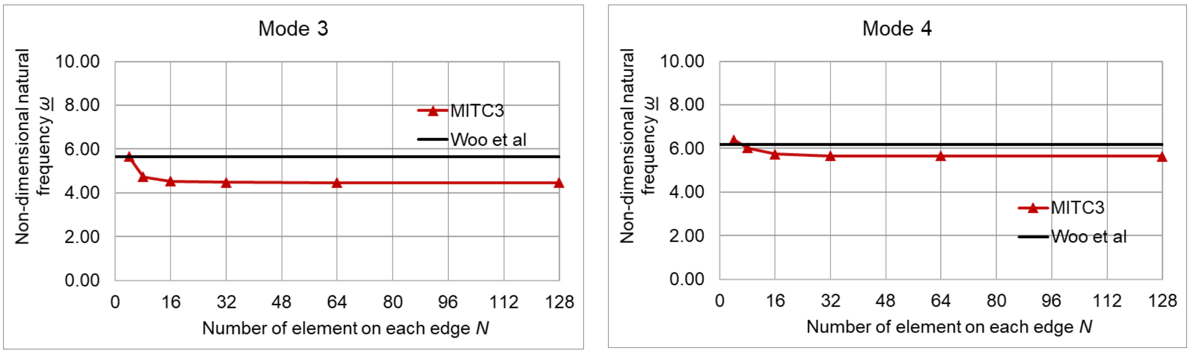


Figure 6. Non-dimensional natural frequency $\underline{\omega}$ of SCSC skew plate (L/h = 5) with skew angle θ = 60.

Figure 6 shows that the MITC3 elements converge to the reference solution in each mode. In fine mesh, skew plate θ = 60˚ has smaller natural frequency than θ = 45˚, the difference is 1,024, it occurs in mode 5, the difference in mode 5 to reference solutions is 22.008%, and mode 1 is 0.133%.

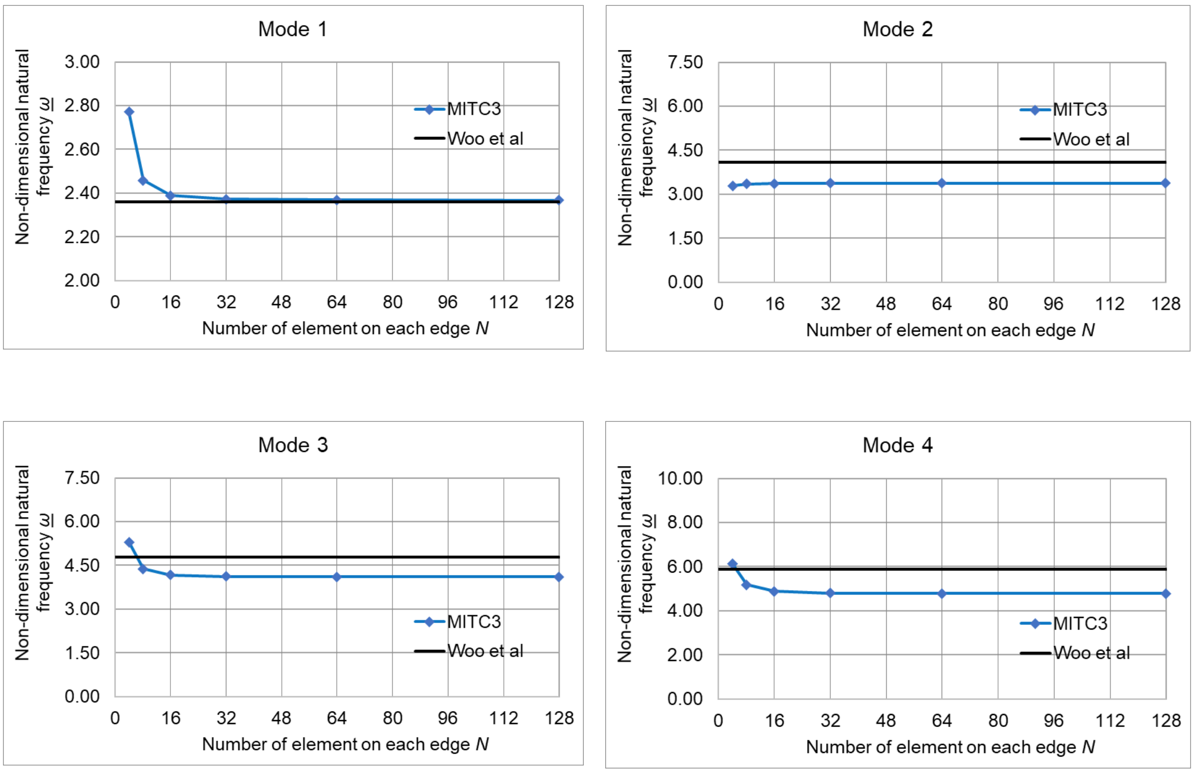
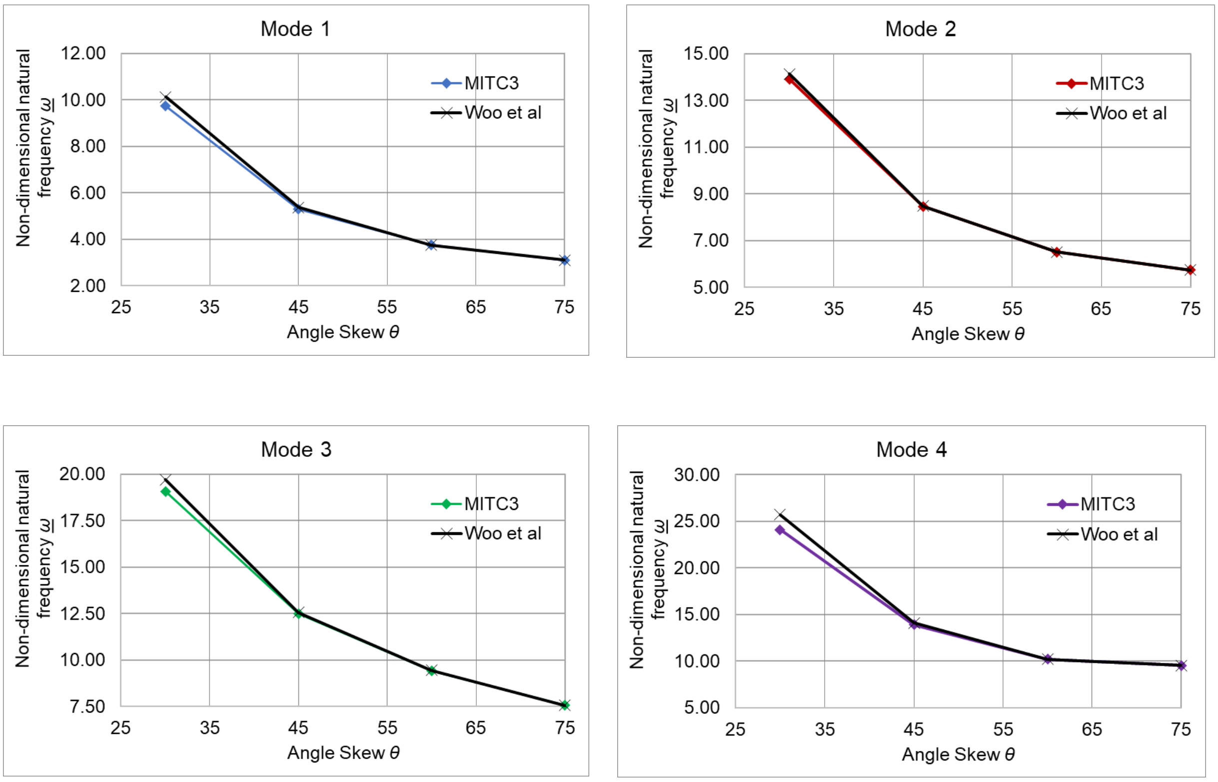


Figure 7. Non-dimensional natural frequency $\underline{\omega}$ of SCSC skew plate (L/h = 5) with skew angle θ = 75.

The analysis depicts that the natural frequency of skew plate θ = 75˚ has a smaller natural frequency than θ = 60˚, and each mode converges to the reference solution in Fig 7. Furthermore, the analysis of the skew plate in (L/h=5) obtained that increasing skew angle θ caused a decrease in the value of natural frequency.

Table 3 The first five non-dimensional natural frequency $\underline{\omega}$ of SCSC skew plate with L/h=1000.

$2N \times N \times 2$	Mode number	Skew Angle θ			
		$\theta = 30^\circ$	$\theta = 45^\circ$	$\theta = 60^\circ$	$\theta = 75^\circ$
128 × 128 × 2	1	9.7504	5.3086	3.7455	3.1115
	2	13.912	8.4688	6.5147	5.7489
	3	19.077	12.497	9.4268	7.5668
	4	24.116	13.887	10.219	9.5380
	5	27.156	17.037	13.965	11.362
Woo et al. **Error! Reference source not found.**]	1	10.124	5.3653	3.7505	3.1082
	2	14.135	8.4885	6.5128	5.7455
	3	19.713	12.565	9.4513	7.5606
	4	25.735	14.084	10.220	9.5315
	5	28.226	17.271	13.980	11.346



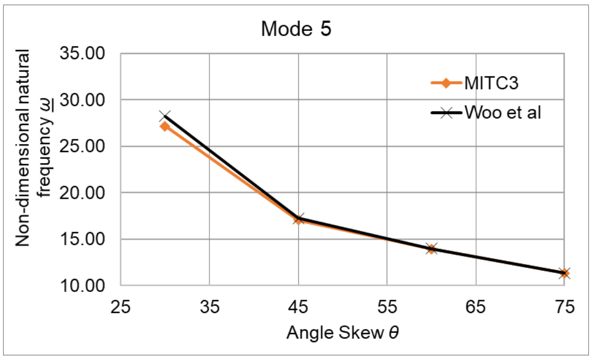


Figure 8. Non-dimensional frequency $\underline{\omega}$ of SCSC skew plate (L/h = 1000) according to skew angle.

The analysis in Table 3 dan Figure 8, MITC3 element convergence to the reference solution. natural frequency with ratio L/h = 1000, mesh 32 × 32 × 2 until mesh 128 × 128 × 2, the difference in mode 1 skew angle θ = 30° to reference solutions is 3.690 – 1.455%, skew angle θ = 60° is 0.892% - 1.057%, skew angle θ = 60° is 0.135% - 2.526%, and skew angle θ = 75° is 0.106% - 4.586%. from it result, it shows increasing skew angle θ caused an decrease in the difference to the reference solution.

In addition, skew plate with ratio L/h = 1000 has bigger natural frequency value than plate L/h = 5, at mesh 128 × 128 × 2 and θ = 30° the difference in mode 1 plate L/h = 1000 to plate L/h = 5 is 4.2894, skew angle θ = 45° is 1.7490, and skew angle θ = 60° is 1.0082, and skew angle θ = 75° is 0.7434.

4. CONCLUSION

Free vibration analysis using one of the FSDT elements was conducted, the analysis was carried out on the skew plate, the result shows MITC3 element convergence to reference solution. Natural frequency value depends on length to width ratios L/h and skews angles θ, increasing the L/h ratio caused an increase in the value of natural frequency, and increasing skew angle caused a decrease in the frequency. Due to the accuracy and simplicity of this element, it conclude MITC3 element can be used to solve the eigenvalue problem.

REFERENCES

[1] P.S. Lee, K.J. Bathe, Development of MITC isotropic triangular shell finite elements, Comput. Struct. 82 (2004) 945–962. https://doi.org/10.1016/j.compstruc.2004.02.004.

[2] E.N. Dvorkin, K.J. Bathe, A continuum mechanics based four-node shell element for general nonlinear analysis, Eng. Comput. 1 (1984) 77–88. https://doi.org/10.1108/eb023562.

[3] K. -J Bathe, E.N. Dvorkin, A formulation of general shell elements—the use of mixed interpolation of tensorial components, Int. J. Numer. Methods Eng. 22 (1986) 697–722. https://doi.org/10.1002/nme.1620220312.

[4] P.S. Lee, H.C. Noh, K.J. Bathe, Insight into 3-node triangular shell finite elements: the effects of element isotropy and mesh patterns, Comput. Struct. 85 (2007) 404–418. https://doi.org/10.1016/j.compstruc.2006.10.006.

[5] Y. Lee, K. Yoon, P.S. Lee, Improving the MITC3 shell finite element by using the Hellinger-Reissner principle, Comput. Struct. 110–111 (2012) 93–106. https://doi.org/10.1016/j.compstruc.

2012.07.004.

[6] Y. Lee, P.S. Lee, K.J. Bathe, The MITC3+ shell element and its performance, Comput. Struct. 138 (2014) 12–23. https://doi.org/10.1016/j.compstruc.2014.02.005.

[7] K.S. Woo, C.H. Hong, P.K. Basu, C.G. Seo, Free vibration of skew Mindlin plates by p -version of F . E . M ., J. Sound Vib. 268 (2003) 637–656. https://doi.org/10.1016/S0022-460X(02)01536-5.

[8] H. Nguyen-Xuan, L. V. Tran, T. Nguyen-Thoi, H.C. Vu-Do, Analysis of functionally graded plates using an edge-based smoothed finite element method, Compos. Struct. 93 (2011) 3019–3039. https://doi.org/10.1016/j.compstruct.2011.04.028.

[9] H. Nguyen-Xuan, L. V. Tran, C.H. Thai, T. Nguyen-Thoi, Analysis of functionally graded plates by an efficient finite element method with node-based strain smoothing, Thin-Walled Struct. 54 (2012) 1–18. https://doi.org/10.1016/j.tws.2012.01.013.

[10] I. Ramu, K.V.G.R. Seshu, M. Venu, Natural frequency of skew FGM plates using finite element method, IOP Conf. Ser. Mater. Sci. Eng. 455 (2018). https://doi.org/10.1088/1757-899X/455/1/012024.

[11] J.N. Reddy, J. N. Reddy Mechanics of Laminated Composite Plates and Shells Theory and Analysis, Second Edition 2003, (2007) 1–855. https://books.google.com/books?id=eeUr_AJiGRcC.